ON GRAPH ISOMORPHISM AND THE PAGERANK ALGORITHM

DISSERTATION

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Abstract

A graph is a key construct for expressing relationships among objects, such as the radio connectivity between nodes contained in an unmanned vehicle swarm. The study of such networks may include ranking nodes based on importance, for example, by applying the PageRank algorithm used in some search engines to order their query responses. The PageRank values correspond to a unique eigenvector typically computed by applying the power method, an iterative technique based on matrix multiplication.

The first new result described herein is a lower bound on the execution time of the PageRank algorithm that is derived by applying standard assumptions to the scaling value and numerical precision used to determine the PageRank vector. The lower bound on the PageRank algorithm's execution time also equals the time needed to compute the coarsest equitable partition, where that partition is the basis of all other results described herein.

The second result establishes that nodes contained in the same block of a coarsest equitable partition must yield equal PageRank values. The third result is an algorithm that eliminates differences in the PageRank values of nodes contained in the same block if the PageRank values are computed using finite-precision arithmetic. The fourth result is an algorithm that reduces the time needed to find the PageRank vector by eliminating certain dot products when any block in the partition contains multiple vertices. The fifth result is an algorithm that further reduces the time required to obtain the PageRank vector of such graphs by applying the quotient matrix induced by the coarsest equitable partition. Each algorithm's complexity is derived with respect to the number of blocks contained in the coarsest equitable partition and compared to the PageRank algorithm's complexity.

These results further existing research in several ways. For instance, the practical lower bound on the PageRank algorithm's execution time was previously only suggested using experimental results. The proof showing vertices contained in the same block of the coarsest equitable partition have equal PageRank values is based on relating dot products and Weisfeiler-Lehman stabilization, which is a much different approach than applied in an existing proof. The existing proof was also extended to show the quotient matrix could be used to reduce the PageRank algorithm's execution time. However, its authors did not develop an algorithm or analyze its execution time bounds. Finally, these results motivate several avenues of future research related to graph isomorphism and linear algebra.

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Chris Augeri

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List of Symbols

Symbol	Meaning
$G=(V,E)$	simple graph, *G*, with a vertex set, *V*, and an edge set, *E*
V	vertex set, $V=\{v_1,v_2,\ldots,v_{n-1},v_n\}$, $1\le i\le n$
E	edge set, $E=\{e_1,e_2,\ldots,e_{m-1},e_m\}$, $m\le n\cdot(n-1)/2$
v_i	arbitrary, but specific, vertex of a vertex set, *V*
e_i, $e_i=\{v_j,v_k\}$	arbitrary, but specific, edge of an edge set, *E*
$\deg(v_i)$	degree of vertex (number of incident edges)
$B=[b_1,b_2,\ldots,b_k]$	disjoint vertex partition, $V=b_1\dot\cup b_2\dot\cup\cdots+\dot\cup b_k$, $b_i\cap b_j=\varnothing$
$\mathbf{x}^{n,1}$	$n\times 1$ vector (lower-case)
$\mathbf{M}^{n,n}$	$n\times n$ matrix (upper-case)
$\mathbf{A}$	adjacency matrix
$\mathbf{D}$	diagonal matrix, (cf. diag below)
$\mathbf{I}$	identity matrix
$\mathbf{P}$	permutation matrix
$\mathbf{1}$, $\mathbf{J}$	matrix whose entries equal one
$\mathbf{0}$, $\mathbf{Z}$	matrix whose entries equal zero
$\mathbf{M}_{i,j}$ or $\mathbf{M}(i,j)$	matrix element (i^{th} row, j^{th} column)
$\mathbf{M}^T$	matrix transpose
$\mathbf{M}_1\cdot\mathbf{M}_2$ or $\mathbf{M}_1\times\mathbf{M}_2$	matrix multiplication (dot product)
$\lambda\,(\Lambda)$	eigenvalue vector (matrix)
$\mathbf{X}$	eigenvectors
$\lVert\mathbf{A}\rVert_p$	vector (matrix) norm, $1\le p\le\infty$
$\mathbf{D}=\operatorname{diag}_i(\mathbf{d})$	construct matrix with $\mathbf{d}$ on i^{th} diagonal, $-(n-1)\le i\le n$
$\mathbf{d}=\operatorname{diag}_i^{-1}(\mathbf{D})$	extract i^{th} diagonal, $-(n-1)\le i\le n$
P	deterministic polynomial complexity
NP	non-deterministic polynomial complexity
$\Omega(n)$	lower bound
$\mathrm{O}(n)$	upper bound
$\Theta(n)$	exact bound
ϕ	permutation, e.g., $\phi=[4,3,5,2,1]\to[d,c,e,b,a]$
$\cong$	isomorphism, e.g., $G_1\cong G_2$ or $\mathbf{A}_1\cong\mathbf{A}_2$
$v_i\to v_j$	permutation mapping of an arbitrary vertex
ω	canonical isomorph, e.g., G_ω or $\mathbf{A}_\omega$

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13. SUPPLEMENTARY NOTES

14. ABSTRACT

Graphs express relationships among objects, such as the radio connectivity among nodes in unmanned vehicle swarms. Some applications may rank a swarm's nodes by their relative importance, for example, using the PageRank algorithm applied in certain search engines to order query responses. The PageRank values of the nodes correspond to a unique eigenvector that can be computed using the power method, an iterative technique based on matrix multiplication. The first result is a practical lower bound on the PageRank algorithm's execution time that is derived by applying assumptions to the PageRank perturbation's scaling value and the PageRank vector's required numerical precision. The second result establishes nodes contained in the same block of the graph's coarsest equitable partition must have equal PageRank values. The third result, the AverageRank algorithm, ensures such nodes are assigned equal PageRank values. The fourth result, the ProductRank algorithm, reduces the time needed to find the PageRank vector by eliminating certain dot products in the power method if the graph's coarsest equitable partition contains blocks composed of multiple vertices. The fifth result, the QuotientRank algorithm, uses a quotient matrix induced by the coarsest equitable partition to further reduce the time needed to compute a swarm's PageRank vector.

15. SUBJECT TERMS

PageRank algorithm, Markov chain, stochastic matrix, stationary probability distribution, dominant eigenvalue, normalized dominant eigenvector graph isomorphism, canonical isomorph, coarsest equitable partition, orbit partition, automorphism group, Weisfeiler-Lehman stabilization quotient graph, quotient matrix, characteristic block matrix, eigenvalue interlacing, eigenvector lifting, matrix multiplication, dot product

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